Appl. Math. Lett. Vol. 4, No. 5, pp. 1–5, 1991
Printed in Great Britain.

0893-9659/91 $3.00 + 0.00

A Note on the Stability of Two-Level Symplectic Schemes

YUHUA WU

Department of Mathematics, University of Texas at Arlington
Computing Center of Academia Sinica

(Received July 1990)

Abstract. An analysis of the stability of symplectic schemes is given for linear Hamiltonian systems. We prove that the symplectic schemes used to solve a linear Hamiltonian system are stable if and only if the linear Hamiltonian system has a positive quadratic conservative quantity. We also consider the stability of general symplectic schemes for linear Hamiltonian systems. The results in this note reveal the relationship existing among Hamiltonian matrices, conservation laws and the stability of their symplectic schemes.

1. Introduction

The *TCS* (time centered symplectic) schemes were first proposed by Feng [1] This kind of scheme is the generalization of time centered Euler schemes with second order accuracy extended to any m'th order accuracy for Hamiltonian systems. The *TCS* schemes have the advantage of preserving all quadratic conservation laws not only for linear Hamiltonian systems but also for general nonlinear Hamiltonian systems [2]. All linear conservation laws are preserved as well [2]. The latter result is true for any symplectic difference schemes constructed by the routine in [1,3]. Actually, the linear symplectic scheme is a conservative scheme, therefore, the stability of symplectic schemes is related to their conservation laws. In this note, we will strengthen the result in [1]. The construction of symplectic schemes is referred to [3,4].

2. Some Preliminary Results on Linear Hamilton Systems

We consider numerical methods for a general Hamiltonian system

$$\frac{dZ}{dt} = BHZ \tag{2.1}$$

where $Z \in R^{2n}$, B is a $2n \times 2n$ antisymmetric and nondegenerate matrix, and H is a $2n \times 2n$ symmetric matrix. According to [5,6], BH is called a B-Hamiltonian matrix, and

$$BH = -B \cdot (BH)^{\mathsf{T}} \cdot B^{-1} \tag{2.2}$$

LEMMA 1. *Suppose B, H are well defined, if λ is an eigenvalue of BH, then $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$ are also eigenvalues of BH. The eigenvalues of BH thus come in 4-tuples.*

The result was given in [6] for a J-Hamiltonian matrix, J being the canonical form of B,

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \tag{2.3}$$

where I_n is an $n \times n$ unit matrix. Obviously, it remains true for a B-Hamiltonian matrix.

LEMMA 2. *The solution of the linear Hamiltonian system (2.1) for each initial value $Z^0 \in R^{2n}$ is bounded for $0 \le t < \infty$ if and only if all elementary divisors of the Hamiltonian matrix BH are linear and $Re\lambda = 0$, where λ is an eigenvalue of BH.*

Lemma 2 can be easily proved by the theory of linear differential equations [7] and Lemma 1. From Lemma 1 and Lemma 2, we get

LEMMA 3. *The solution of (2.1) is bounded for $0 \le t < \infty$ if and only if Hamiltonian system (2.1) has a positive quadratic conservative quantity.*

COROLLARY 3. *If H is a positive matrix, then the solution of system (2.1) is bounded.*

DEFINITION: Let $|| \cdot ||$ denote the well defined norm on R^{2n}. If there exists a positive constant C, such that

$$||Z(t)|| \le C||Z^0||, \qquad for\ 0 \le t < \infty, \qquad \text{for any } Z^0 \in R^{2n} \tag{2.4}$$

where $Z(t)$ is the solution of the Hamiltonian system

$$\begin{aligned} \frac{dZ}{dt} &= BHZ \\ Z|_{t=0} &= Z^0 \in R^{2n} \end{aligned} \tag{2.5}$$

then the system (2.9) is called stable.

From Lemma 3, we conclude that a linear Hamiltonian system is stable if and only if the linear system has a positive quadratic conservative quantity, which is a classical result [7]. Here we need Lemmas 1 through 3 to study the stability of the linear symplectic schemes.

3. ANALYSIS OF THE STABILITY OF LINEAR SYMPLECTIC SCHEME

Consider the TCS scheme of the $2m$'th order accuracy for a general Hamiltonian system [2],

$$\frac{dZ}{dt} = Bh_z(Z), \qquad Z \in R^{2n}.$$

The TCS scheme is written as follows:

$$Z^{i+1} - Z^i = \sum_{i=1}^{m} B \frac{\partial S^{(2i-1)}(\overline{W})}{\partial \overline{W}} \tau^{(2i-1)} \tag{3.1}$$

where

$$S^{(1)} = h(z)$$

and

$$S^{(2i-1)}(\overline{W}) = \sum_{1 \le m \le 2i-1} \frac{1}{(2i-1)!} \frac{1}{m!} \sum_{i_1, \dots, i_m = 1}^{2n} h_{z_{i_1} \cdots z_{i_1}}(\overline{W}).$$

$$\sum_{\substack{k_1 + \cdots + k_m \\ k_i \\ k_i \text{ is odd}}} \frac{1}{2^m} \left(BS_{\overline{w}}^{(k_1)}\right)_{i_1} \cdots \left(BS_{\overline{w}}^{(k_m)}\right)_{i_m} \tag{3.2}$$

$$i = 2, 3, \dots,$$

where τ is the time stepsize, $\overline{W} = \frac{1}{2}(z^{i+1} + z^i)$, and $h = h(Z)$ is an analytic function in Z.

When h is a quadratic function, scheme (3.2) can be rewritten as follows [1]:

$$Z^{k+1} = \Psi_{2m-1}^{-1}(-\tau BH)\Psi_{2m-1}(\tau BH)Z^k \tag{3.3}$$

where

$$\Psi_{2m-1}(\tau BH) = I + \sum_{k=1}^{m} \alpha_{2k-1} \left(\frac{\tau}{2} BH\right)^{2k-1} \tag{3.4}$$

$$\alpha_{2k-1} = 2^{2k} \left(2^{2k} - 1\right) B_{2k}/(2k)! \tag{3.5}$$

B_{2k} is a Bernoulli number.

DEFINITION: The linear difference scheme

$$Z^{k+1} = \Phi(\tau) Z^k \tag{3.6}$$

is called L-stable, if there exists a positive constant C, such that the solution of (3.6) satisfies

$$||Z(t)|| \leq C||Z^0||, \qquad \text{for all n}, \qquad \forall Z^0 \in R^{2n} \tag{3.7}$$

According to [1,2], we know that the TCS schemes preserve all quadratic conservative quantities, so we get the following:

LEMMA 5. *The TCS scheme used for solving a linear stable Hamiltonian system is L-stable.*

LEMMA 6. *All eigenvalues $\{\lambda\}$ of a symplectic matrix*

$$S = \Psi_{2m-1}^{-1}(-\tau BH)\Psi_{2m-1}(\tau BH) \tag{3.8}$$

have linear elementary divisors and $|\lambda| = 1$ if and only if the eigenvalues $\{\lambda\}$ of the Hamiltonian matrix BH have linear elementary divisors and $Re\, \lambda = 0$.

Lemma 6 can be easily proved by means of Lemma 1 to Lemma 3. According to [8], the eigenvalues of a symplectic matrix come in 4-tuples λ, $\overline{\lambda}$, $1/\lambda$, $1/\overline{\lambda}$. Therefore, the linear symplectic scheme

$$Z^{k+1} = K_\tau Z^k \qquad \text{and} \qquad K_\tau^{\top} J K_\tau = J \tag{3.9}$$

where K_τ is named as the symplectic matrix, is L-stable if and only if any eigenvalue of symplectic matrix K_τ has linear elementary divisors and is unimodular. Hence our result follows.

THEOREM A. *The TCS schemes used to solve a linear Hamiltonian system is L-stable if and only if the linear Hamiltonian system has a positive quadratic conservative quantity.*

THEOREM B. *The linear symplectic scheme (3.9) is L-stable if and only if it has a positive quadratic conservative quantity.*

PROOF: The linear symplectic scheme (3.9) is L-stable if and only if the eigenvalues of K_τ have linear elementary divisors and are unimodular. Because the eigenvalues of K_τ come in 4-tuples λ, $\overline{\lambda}$, $1/\lambda$, $1/\overline{\lambda}$. So according to matrix theory, we can find a real nonsingular matrix P, such that

$$P^{-1}K_\tau P = \begin{bmatrix} \Lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & \Lambda_n \end{bmatrix}_{n\times n} \tag{3.10}$$

where

$$\Lambda_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$$

and

$$\alpha_i^2 + \beta_i^2 = 1, \; i = 1, \dots, n\,. \tag{3.11}$$

Therefore, we transform Z to W such that

$$W^{k+1} = P^{-1}K_\tau P W^k\,. \tag{3.12}$$

From (3.12), we get

$$\begin{aligned} W_{2l-1}^{k+1} &= \alpha_l W_{2l-1}^k + \beta_l W_{2l}^k \\ W_{2l}^{k+1} &= -\beta_l W_{2l-1}^k + \alpha_l W_{2l}^k \\ &l = 1, 2, \ldots, n. \end{aligned} \tag{3.13}$$

Hence

$$\left(W_{2l-1}^{k+1}\right)^2 + \left(W_{2l}^{k+1}\right)^2 = \left(W_{2l-1}^k\right)^2 + \left(W_{2l}^k\right)^2. \tag{3.14}$$

Namely, there is a positive quadratic conservative quantity:

$$\left(P^{-1}Z^k\right)^T \left(P^{-1}Z^k\right) = \text{constant}. \tag{3.15}$$

The matrix

$$Q = \left(P^{-1}\right)^T P^{-1}$$

is a positive matrix, and

$$||Q|| \leq C_1$$

where C_1 is a constant independent of the iteration step k. The theorem is proved.

From Theorem B, we conclude that a better scheme for approximating a linear stable Hamiltonian system is the symplectic scheme with a positive quadratic conservative quantity. Other methods (nonsymplectic schemes, especially explicit schemes) are much more likely to be L-unstable for stable linear Hamiltonian systems, such as forward and backward one step Euler methods, and one step methods constructed by truncating Taylor series of the flow of linear Hamiltonian systems. This is because the necessary condition for the stability of a linear Hamiltonian system $Re\,\lambda = 0$, where λ is an eigenvalue of BH. A scheme which is only A-stable is not suitable for numerical computation of Hamiltonian systems. It is very interesting that some implicit Runge-Kutta schemes [9] are symplectic when applied to linear Hamiltonian systems and also have quadratic conservation laws. It is essential that a scheme has a positive quadratic conservation law for stable linear Hamiltonian systems. The symplectic approximation may provide this property. We can prove that any consistent symplectic scheme approximating a linear strongly stable Hamiltonian system is L-stable.

THEOREM C. *Suppose a symplectic scheme is convergent for a linear Hamiltonian system, and if this scheme is L-stable, then the linear Hamiltonian system has a positive conservative quantity.*

4. Some Remarks

REMARK 1: If the TCS scheme is replaced by

$$Z^{k+1} = \Psi(\tau BH)Z^k$$

where $\Psi(\lambda)$ is analytic function with real coefficients [1,4] in a neighborhood D of $\lambda = 0$, and satisfies

$$\Psi(\lambda)\Psi(-\lambda) = 1$$

in D, $\Psi_\lambda(0) \neq 0$, the result of theorem A is true for this type of symplectic schemes for linear Hamiltonian systems.

REMARK 2: A sufficient criterion of instability is given in the following [10]. For a symmetric matrix P, such that

$$P^{-1}HP = \begin{bmatrix} K & A^T \\ A & N \end{bmatrix}$$

and $K = K^T > 0$, $N = N^T < 0$, we have $Re\,\lambda \neq 0$, where λ is the eigenvalue of J-Hamiltonian matrix JH. Therefore, we can ensure that the TCS scheme approximating this linear Hamiltonian system is L-unstable by Theorem A. In other words, if $Re\,\lambda = 0$, the symmetric matrices N, K do not have the definiteness with opposite sign.

REMARK 3: For a two-dimensional linear Hamiltonian system, the criterion for stability is much simpler. The linear autonomous Hamiltonian system in two dimensions is L-stable if and only if the trace of Hamiltonian matrix BH is zero, namely, H is a positive matrix, so the TCS scheme has the positive quadratic conservation laws.

A general linear symplectic scheme

$$Z^{k+1} = K_\tau Z^k$$

in two dimensions is L-stable if $|\mathrm{tr}\, K_\tau| < 2$, and unstable if $|tr\, K_\tau| > 2$. If $tr\, K_\tau = 2$, or -2, the scheme is L-stable provided the matrix is I or $-I$.

REFERENCES

1. K. Feng, *J. Compt. Math,* **4:3**, 279–288 (1986).
2. Wu Yuhua, Symplectic transformation and symplectic schemes, preprint, (1987).
3. K. Feng and M. Qin, The symplectic methods for the computation of Hamiltonian equations, *Lecture Notes in Mathematics* No. 1297, Springer-Verlag.
4. H. Wu and K. Feng, Symplectic difference schemes for canonical Hamiltonian systems, *Third Annual Meeting of Chinese Computational Mathematics Society*, Bejiing, (1985).
5. C. Van Loan, *Linear Algebra and Its Application*, Vol. 61, pp. 233–251, (1984).
6. A.J. Laube and K. Meyer, *Celestial Mechanics*, Vol. 9, pp. 213–238, (1974).
7. V.A. Yakubovich and V.M. Starzhinskii, *Linear Differential Equation with Periodic Coefficients*, John Wiley and Sons, New York, (1975).
8. V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, (1980).
9. H.J. Stetter, *Analysis of Discretization Methoda for Ordinary Differential Equations*, Springer-Verlag, Berlin, (1973).
10. C. Paige and C. Van Loan, *Linear Algebra and Its Application*, Vol. 41, pp. 11–32, (1981).

Department of Mathematics, Box 19408, University of Texas at Arlington, TX 76019 U.S.A.
Computing Center Academia Sinica, P.R. China